Math 33A Worksheet Week 9 Solutions

TA: Emil Geisler and Caleb Partin

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Exercise 1.
Recall that we can compute the determinant of a $n \times n$ matrix through **Cofactor Expansion**. If A is the matrix we are trying to compute the determinant of, then we can let A_{ij} denote the $(n-1) \times (n-1)$ matrix formed by removing the ith row and jth column from A. Then for some choice of column j

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

or some choice of row i

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

That is, we can compute the determinant by summing over the cofactors for any row and column. Use these formulas to compute the determinant of the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 \end{bmatrix}$

(a) $\det\left(\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}\right) = 1\cdot 6 - 2\cdot 3 = 0$

(b) $\det\left(\begin{bmatrix}2 & 5 & 7\\ 0 & 11 & 7\\ 0 & 0 & 5\end{bmatrix}\right) = 2\cdot\det\left(\begin{bmatrix}11 & 7\\ 0 & 5\end{bmatrix}\right) = 2\cdot(11\cdot 5 + 0\cdot 7) = 110$

(c) $\det\left(\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}\right) = 1\cdot\det\left(\begin{bmatrix}2 & 2\\ 3 & 3\end{bmatrix}\right) - 1\cdot\det\left(\begin{bmatrix}2 & 2\\ 3 & 3\end{bmatrix}\right) + 1\cdot\det\left(\begin{bmatrix}2 & 2\\ 3 & 3\end{bmatrix}\right) = 2\cdot 3 - 2\cdot 3 = 0$

(d)

$$\det\left(\begin{bmatrix}1 & 2 & -1 & 3 & 1 & 1\\ 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & -1\\ 0 & 1 & 0 & 0 & 2 & -1\end{bmatrix}\right) = \det\left(\begin{bmatrix}0 & 2 & 0 & 0 & 0\\ 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & -1\\ 1 & 0 & 0 & 2 & -1\end{bmatrix}\right) = \det\left(\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix}\right)$$

$$= 2\cdot\det\left(\begin{bmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 1 & 1 & -1\end{bmatrix}\right) = 2\cdot 3\cdot\det\left(\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right) = 6\cdot(-1-1) = -12$$

Exercise 2. One important application of determinants is that they help us determine invertibility of a matrix. Use the determinant to determine for which values λ the following matrices are invertible:

(a) $\begin{bmatrix} \lambda & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}$

(d) $\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 1 \\ 0 & -1 & 6-\lambda \end{bmatrix}$

(a) $\det\left(\begin{bmatrix} \lambda & 2 \\ 3 & 4 \end{bmatrix}\right) = 4\lambda - 6$

$4\lambda - 6 = 0 \implies \lambda = \frac{3}{2}$ is the only value when this matrix is not invertible.

(b) $\det\left(\begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix}\right) = (\lambda-1)(\lambda-\lambda^2) = (\lambda-1)^2\lambda$ So this matrix is not invertible when $\lambda = 0, 1$

(c) $\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}\right) = (1-\lambda)(4-\lambda)$

So this matrix is not invertible when $\lambda = 1, 4$

(d) $\det\left(\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 1 \\ 0 & -1 & 6-\lambda \end{bmatrix}\right) = (3-\lambda)[(4-\lambda)(6-\lambda)+1] = (3-\lambda)(\lambda^2 - 10\lambda + 25) =$ $(3-\lambda)(\lambda-5)^2$

So this matrix is not invertible when $\lambda = 3, 5$

Exercise 3. Determine whether the following are true or false:

(a) $\det(A+B) = \det(A) + \det(B)$ for any two $n \times n$ matrices A and B.

(b) If B is the rref of A, then $\det(B) = \det(A)$.

(c) There exists a 3×3 matrix A with real valued entries such that $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(d) If A is an orthogonal matrix, then $\det(A) = \pm 1$.

(a) False. Here's a counter-example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1 + 1 \neq \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$$

(b) False, scaling a matrix or swapping rows changes the determinant!
(c) False. Such a matrix A would have $\det(A^2) = -1 \implies \det(A) = \sqrt{-1}$, however, if A is composed of all real entries, this cannot be the case.
(d) True. If A is orthogonal, we know that $A^T = A^{-1}$ and thus we have that $\det(A) = \det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)} \implies \det(A)^2 = 1 \implies \det(A) = \pm 1$

Exercise 4. Write down the relationship between $\det(B)$ and $\det(A)$ for the following scenarios (**Challenge:** Can you prove any or all of them?)

(a) If B is obtained by multiplying a row of A by a scalar k.

(b) If B is obtained by swapping two rows of A.

(c) If B is obtained by adding a multiple of one row of A to another row of A.

(d) $B = A^T$

(e) If A is invertible, $B = A^{-1}$

Send us an email or come to office hours if you want to discuss how to prove any of these!
(a) $\det(B) = k\det(A)$
(b) $\det(B) = -\det(A)$
(c) $\det(B) = \det(A)$
(d) $\det(B) = \det(A)$
(e) $\det(B) = \frac{1}{\det(A)}$